TWO STAGES OF MATHEMATICS CONCEPT LEARNING: ADDITIONAL APPLICATIONS IN ANALYSIS OF STUDENT LEARNING

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Tzur and Simon (2004) postulated two stages of concept development, participatory and anticipatory. The distinction between the two stages was exemplified by what they termed "the next-day phenomenon" in which learners who could solve a task one day in the context of the activity through which they made the abstraction, could not solve what seemed to be the same task in a subsequent lesson when the students were not engaged in or thinking about the activity. Here we expand the application of this theoretical distinction by providing two examples of use of the distinction in analyses of data segments that are different from the next-day phenomenon.

BACKGROUND

As part of a program of research on conceptual learning of mathematics, Tzur and Simon (2004) postulated two stages of development in learning a mathematical concept: participatory and anticipatory. To illustrate the stage distinction, Tzur and Simon (2004) gave the following example of what they called the "next day phenomenon":

> Consider a teacher who engaged learners for a few lessons in partitioning paper strips to create unit fractions. Toward the end of this hands-on activity, the learners were able to answer questions such as, "Which is larger, 1/6 or 1/8?" The teacher required the learners to explain their answers and most learners could clearly demonstrate with their strips and argue that 1/8 must be smaller than 1/6, because the strip showing eighths was cut into more pieces; so each piece had to be smaller. The next day, … the teacher begins the lesson by attempting to review the ideas generated by the learners during the paper-strip activity. The teacher writes two fractions on the board, '1/7' and '1/5,' and asks which one is larger. To the teacher's surprise, most of the learners claim that 1/7 is larger because 7 is larger than 5. The teacher wonders how learners can "lose" overnight what they learned the day before. Intending to revisit the hands-on experience, the teacher asks the learners to take out their paper strips to set up the problem. Soon after the learners begin manipulating the paper strips, and without completing a paper-strip enactment of the problem posed, many learners, who had earlier claimed that 1/7 was larger, raise their hands to explain how they know that 1/5 is larger than 1/7. (p. 288-289)

Tzur and Simon (2004) argued that it is not a case of learners forgetting what they learned the day before, but rather that there are two distinct stages of abstraction that occur as a learner is developing a new mathematical concept. In the first stage, labelled *participatory*, learners develop an anticipation based on engagement in a particular activity. That is, through engaging in the activity, they develop knowledge of a

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mathematical relationship *and* no longer need to carry out the activity to determine the result. Furthermore, the learner can justify and explain the logical necessity of the result. However, at this first (participatory) stage, this anticipation is limited. Learners have not yet learned to call upon the abstraction (anticipation) when they are not involved with or thinking about the activity through which it was learned. If the learners are presented with a seemingly similar task the next day, outside of the context of the activity through which the anticipation was learned, they are not able to call on the relevant (from the observer's perspective) anticipation. However, in the second stage of abstraction, labelled *anticipatory*, the learner is able to call upon the learned anticipation even when not engaged in the activity through which it was learned.

The distinction between these stages of understanding implies that (for the learner) the next day's task was not the same as the prior day's task, which the learners *were* able to solve, even if the tasks were word-for-word the same. The same question posed in the context of the paper-folding activity was not the same as the question posed unconnected to the paper folding activity. Essentially, the first asked for an anticipation of the results of paper folding (partitioning), whereas the second was a more general question about fractions with no hint of how to approach it. The distinction also suggests that we define *task* not as just the written or oral articulation of the task. Rather the task is in part defined by its place in a sequence of tasks and by the tools available or given to the learner with which to work.

OUR RESEARCH PROGRAM

Before we present data to demonstrate the usefulness of the stage distinction in a long-term teaching experiment, we provide some background on the research program and current project from which these examples are drawn.

We have been engaged in a research program, "Learning through Activity (LTA)" (Simon, 2013; Simon et al., 2010) that builds on Piaget's (2001) theoretical construct of reflective abstraction. The aims of the program are to both explain learners' development of mathematical concepts (a detailed examination of reflective abstraction in mathematics learning) and develop principles for promoting mathematical concept learning. As such, it is a research program intended to develop integrated theory on aspects of mathematics learning and pedagogy.

This paper is based on research conducted during the second and third years of the five-year Measurement Approach to Rational Number (MARN) project[1]. The project is focused on two goals: (1) increasing understanding of how learners learn through their mathematical activity (LTA), and (2) understanding how learners can effectively develop fraction and ratio concepts through activities grounded in measurement. In this article, we present data from the second phase of the project (years 2 and 3), five

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one-on-one teaching experiments on fraction and ratio concepts. This phase involved developing and implementing task sequences for fraction and ratio learning and modifying those trajectories based on ongoing analyses. More in depth retrospective analyses followed. The data in this paper comes from a one-on-one teaching experiment with Kylie, during her fourth and fifth grade years (fall, 2011 through spring, 2013) in two one-hour sessions per week.

Much of the students work was done in the software environment, JavaBars (Biddlecomb & Olive, 2000). In JavaBars, quantities are represented by rectangles of different lengths. The bars can be partitioned and bars and parts of bars can be iterated.[2] Although the examples do not contain actual use of JavaBars, the conversations refer to that work.

APPLICATIONS OF THE STAGE DISTINCTION IN RETROSPECTIVE ANALYSES

The participatory-anticipatory distinction is a key theoretical construct in our analysis of data. As demonstrated above, it explains the seeming inconsistency of performance from one session to another (the next-day phenomenon). Because of the frequency of data of this type, this alone is an important function of the construct. However it is not just in comparing learner work from different sessions that the construct has proved useful. Examples 3 and 4 below demonstrate the usefulness of the stage distinction for explaining data that do *not* follow the form of the next-day phenomenon.

Example 1

After not working with Kylie for more than 4 months (mid-May to late September), we started our second year of work by doing an assessment. We discuss here two of the questions that emerged from the analysis of the data generated and how the participatory-anticipatory distinction was useful in postulating answers to those questions.

Midway through the assessment, Kylie was given the following tasks in succession:

Task 1.1: This bar is three-sevenths of a unit long. I repeat it one hundred times, how long is my new bar?

Kylie said three seven-hundredths and then changed her answer to three hundred sevenths. When she was asked for justification, she changed her answer to three hundred seven-hundredths. She could not provide justification for any of the calculations.

Task 1.2: This bar is two-fifths of a unit long. I repeat it four times, how long is my new bar?

[2] Frank Iannucci modified JavaBars for us to include an “iterate” button that creates a new bar the specified number of iterations of the original bar.

Kylie once again multiplied both the numerator and denominator by four resulting in eight twentieths. She was not able to justify her solution and did not seem to have confidence in it.

Task 1.3: This bar is one-sixth of a unit long. I repeat it eleven times, how long is my new bar?

S: Eleven-sixths

R: Eleven-sixths?

S: Yeah.

R: Okay, convince me.

S: Well, I repeated it that many...Oh I know what the other one is [referring to the previous task].

R: Yeah, what?

S: It's eight-fifths.

R: Okay, are you sure?

S: Yes!

Task 1.4: This bar is fourth-ninths of a unit long. I repeat it twenty-five times.

R: What's that one?

S: It's uh...I know, four times twenty-five is a hundred-ninths.

In our analysis, we were initially puzzled by the data. Why could she do Task 1.3 correctly, but not 1.1 and 1.2? Why was she able to do Task 1.2 (and 1.4) after solving 1.3, but not before? The explanation we settled on is the following. In order for Kylie to be able to figure out the bar that would be produced by repeating a two-fifths-unit bar four times, she would have to think about the two-fifths- unit bar as being the result of iterating a one-fifth-unit bars two times. She likely was thinking with a part-whole scheme. Kylie had previously demonstrated that she could *make* two-fifths by partitioning a unit into 5 parts, pulling out 1 part and iterating it twice. However, the data seems to show that Kylie did not have an anticipatory-stage conception of a non-unit fraction as the result of an iteration of a unit fraction. That is, she did not think to call on that idea in the context of Tasks 1.1 and 1.2. Whether or not a student can do so is Steffe and Olive's (2010) distinction between a partitive fraction scheme and an iterative fraction scheme. However, when she was asked the result of stringing together unit fractions in Task 1.3, it caused her to engage in the activity of creating a non-unit fraction through iteration. Following that task, she was able to solve Task 1.2 and Task 1.4. These tasks were now participatory tasks; that is, in the context of thinking about iteration of a unit fraction to make a non-unit fraction, she was able to now think about the non-unit fraction (two-fifths) as the result of iterating a unit fraction (one-fifth). That allowed her to solve the tasks involving the iteration of a non-unit fraction (Tasks 1.2 and 1.4). One might say that she was at the participatory stage of an iterative fraction scheme.

Example 2

In this session below, Kylie had been learning concepts of ratio. The tasks were designed to develop an abstraction of the multiplicative relationship between the two quantities and to understand the invariance of that relationship.

Task 2.1: One of the giant's steps is equal to six of Kylie's steps. If the giant walks 84 miles, how far would Kylie go in the same number of steps?

Kylie needed to anticipate that the relationship between the giant's steps and her steps was multiplicative and that she could use this relationship to determine the number of miles she walked, making use of the invariance of that relationship. (Note, the use of two different units of length in the task, steps and miles, with no conversion factor provided, is meant to pre-empt the task being solved with a simple application of a per-one strategy or a build-up strategy).

K: What's eighty-four divided by six?

I: Fourteen

K: I walk fourteen miles.

I: Why did you divide?

K: I know for each step the giant takes, I take six. … Every time the giant walks eighty-four miles, I walk fourteen. It's like one-sixth of eighty-four.

I: How is that related to you and the giant?

K: Not sure. Oh yeah, every time he takes a step, I have to take six, so if I only take one step that's only one-sixth of his step.

In the task above, Kylie shows an anticipation of the invariance of the multiplicative relationship between how far she walks and how far the giant walks. One can think about this anticipation as having two interrelated parts, comparing the two quantities multiplicatively and knowing that that relationship is invariant across any distance travelled.

Later in the session, we gave her a slightly different task for which Kylie did not exhibit the same anticipation.

Task 2.2: Forty-two of Kylie's steps are equal to twenty-one of Marty's steps, If Kylie walks twenty-five miles, how far does Marty walk taking the same number of steps?

K: One hundred miles. Wait! I have to find out how many steps I take when Marty takes one step. [Note: If the researcher had allowed Kylie to proceed in this manner, she would likely have set herself up for a similar solution to the task above. However, his next question is what changed the task for Kylie.]

I: Can you look at these numbers and tell me the answer?

K: Seven? I thought forty-two, six times seven, and forty-two divided by six is seven.

I: And the twenty-one doesn't matter?

K: Ohhh, yes it does.

I: If I tell you twenty-one of my steps is forty-two of your steps, what do you know about our steps?

K: Yours is bigger

I: How much bigger?

K: Twenty-one steps bigger

The data presented were puzzling. How could we account for the difference in Kylie's anticipation in these two situations? The participatory-anticipatory distinction proved to be useful in allowing us to generate a hypothesis. Based on prior analyses of Kylie's learning and the data described here, we made the following inferences. Kylie had had significant experience considering units and partial units, which was the basis for how she developed her fraction concept: a partial unit (unit fraction) is a part that iterates a certain number of times to make a unit. In Task 2.1, the given information that six of Kylie's steps are equivalent to one giant step likely cued the activity of iterating a partial unit to make a unit. Since, for Kylie, the relationship between a partial unit and the unit is a multiplicative relationship, she was able to anticipate the multiplicative relationship between her steps and the giant's steps and then use her understanding of the invariance of this relationship to determine the number of miles she would walk.[3]

In Task 2.2, Kylie was about to convert this task to the form of the previous task (i.e., the number of Kylie's steps in the larger step) in order to solve it in the same way. However, the researcher prevented that approach. As a result, Kylie was not able to mentally iterate one of her steps to make the larger step, so she did not think to use this relationship between partial units and units and thus did not consider the multiplicative relationship between the quantities. Once again, we see evidence of knowledge that is only at the participatory stage. The task as constrained by the researcher's follow-up question was an anticipatory task. Because it did not cue the activity of iterating a partial unit for Kylie, she was unable to use her anticipation that was tied to that activity (i.e., iterating her step to make a giant step). In lieu of thinking about iteration of a partial unit, she thought only about the additive comparison.

In these two examples of using the participatory-anticipatory distinction to explain puzzling data, we have demonstrated its usefulness in situations that are not of the form of the next-day phenomenon. In the next day phenomenon, the same task can be either a participatory task or an anticipatory task, depending on what came before it (i.e., whether the learner is thinking about the key activity). In this last example, it was not the order of the tasks that was crucial, but rather the extent to which the task evoked thinking about the key activity. The participatory-anticipatory distinction structured

[3] Note, we are not offering an explanation of the basis for her anticipation of the multiplicative invariance for two reasons. First, it is not critical for making our intended point here, and second, we do not yet fully understand it.

our examination of the data to focus on what activity might have afforded the anticipation in Task 2.1 and how Task 2.2 might have not afforded the same access to that activity and the related anticipation.

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